DOE/ER/03077-249

Courant Mathematics and
Computing Laboratory

U.S. Department of Energy

Elementary Waves and Riemann Solutions: Their Theory and Their Role in Science

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Research and Development Report

Supported by the Applied Mathematical Sciences
subprogram of the Office of Energy Research,
U.S. Department of Energy under
Contract DE-AC02-76ER03077

Mathematics and Computers

May 1985

NEW YORK UNIVERSITY

Printed in U.S.A.

Available from

National Technical Information Service
U.S. Department of Commerce
5285 Port Royal Road
Springfield, VA 22161

DOE/ER/03077-249
UC-32
Mathematics and Computers

Courant Mathematics and Computing Laboratory

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UNCLASSIFIED

CONTENTS

ELEMENTARY WAVES AND RIEMANN SOLUTIONS: THEIR THEORY AND THEIR ROLE IN SCIENCE

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ABSTRACT

Elementary waves and Riemann solutions are solutions of a nonlinear conservation law which are distinguished by symmetry properties. In this paper we discuss the theory of elementary waves and Riemann solutions with an emphasis on general concepts and open problems. The importance of these solutions to science will also be discussed.

1. Supported in part by the National Science Foundation, grant DMS - 831229.
2. Supported in part by the Applied Mathematical Sciences subprogram of the Office of Energy Research, U. S. Department of Energy, under contract DE-AC02-76ER03077
3. Supported in part by the Army Research Office, grant DAAG29-83-K-0007

1. Introduction

Elementary waves and Riemann solutions are solutions of a nonlinear system of conservation laws

$$u_t + \nabla \cdot \vec{f}(u) = 0 \tag{1.1}$$

which are distinguished by symmetry properties. We take (1.1) to be an $n \times n$ system, considered in d space variables. A Riemann solution is a solution u of (1.1) which is invariant under the scale transformation $\vec{x}, t \to s\vec{x}, st$, $s > 0$. An elementary wave with velocity $\vec{\sigma} \in R^d$ is a Riemann solution which is invariant under the group action $\vec{x}, t \to \vec{x} + \vec{\sigma}\tau, t + \tau$. Equivalently these solutions are transformed into a stationary solution ($u_t = 0$) by a Galilean transformation $\vec{x}, t \to \xi, t = \vec{x} - \vec{\sigma}t, t$. The elementary waves and Riemann solutions are said to have co-dimension j if they are independent of all but j of their space coordinates.

Let $\vec{A} = \dfrac{\partial \vec{f}}{\partial u}$ be the Jacobean matrix of $\vec{f}$. Then (1.1) is hyperbolic if the $n \times n$ matrix $\vec{A} \cdot \xi$ has real eigenvalues $\lambda_1, \cdots, \lambda_n$ for each $\xi \in R^d$ and it is strictly hyperbolic if the λ_j are furthermore distinct. Since $\vec{A} = \vec{A}(u)$ and $\lambda_j = \lambda_j(u,\xi)$ depend on the point u in the state space (which is some domain contained in R^n), we can speak of both local properties (such as local hyperbolicity) and global properties. The local properties are then valid for all u in some neighborhood of some u_o while the global properties are valid for all u in the state space. The eigenvalues $\lambda(u,\xi)$ are the characteristic speeds or the wave speeds for propagation of disturbances of infinitesimal size in the direction ξ. This is the same as saying that they are the speeds of the system

$$w_t + (\vec{A} \cdot \xi)\,(\xi \cdot \nabla) w = 0 , \tag{1.2}$$

which is the linearization of (1.1) about the constant solution $u(\vec{x},t) \equiv u \in R_n$. Similarly the right eigenvectors $e_j = e_j(u,\xi)$ of $\vec{A}(u) \cdot \xi$ are the normal modes for the linearized system (1.2). See also [8,9] for a further discussion of these concepts.

2. Solution Singularities

The necessary occurance of solution singularities is the most interesting phenomenon associated with nonlinear conservation laws such as (1.1). It is remarkable that smooth data does not force and usually does not allow smooth solutions. Jump discontinuities form spontaneously in the solutions to (1.1). The classification of the allowed discontinuities and their interactions is an important and probably central aspect of the qualitative understanding of general solutions.

The elementary waves are an idealized form of the solution discontinuities. The idealization is to perform an infinite limit of scale transformations in a neighborhood of a discontinuity. The result of this limit is a scale invariant solution of (1.1) with the same discontinuity. The discontinuity propagates with a definite velocity without change of form, and thus is an elementary wave. A simple jump discontinuity has co-dimension one, and the generic intersection point of j jump discontinuities has co-dimension j and defines a co-dimension j elementary wave. Usually more than the minumum number j of jump discontinuities are involved in a co-dimension j elementary wave.

The interaction of discontinuities is an isolated event in time. The idealization of the infinite scaling limit can still be performed, and yields a scale invariant solution. However the result is a discontinuity which occurs as a single instant of time, and which does not preserve its form by moving coherently with a definite velocity. Thus we have a Riemann solution rather than an elementary wave. The corresponding Cauchy data is known as a Riemann problem, and it defines in idealized form the interaction of elementary waves.

The interaction of discontinuities can be thought of as a scattering problem, and in this context, the incoming wave operator is the operation of bringing to a common point in space and time several elementary waves. This data, at $t = t_o$, is a Riemann problem and its solution, the Riemann solution, for $t \geq t_o$, is the outgoing wave operator of this scattering problem.

There is a very simple and appealing picture to describe qualitatively these Riemann solutions. In fact a Riemann solution of co-dimension j will consist of a number of elementary waves of co-dimension i, moving apart (scattering) each with their own distinct velocity. These elementary waves are in turn connected by strings, surfaces, ..., i. e. by elementary waves of lower co-dimension. A rather satisfactory analysis of the scalar Riemann problem in two space dimensions ($n = 1, d = 2$) has been obtained [10,16,17,26] which supports this picture. As discussed in [8], the phenomena of reverberating [23] or recombining [17] waves can give rise to an infinite or unbounded number of elementary waves (loss or near loss of piecewise smoothness) if the order, n, or the number of inflections points in a single mode is not rather small, depending on d.

The same idealization of an infinite scaling limit can be used to see that the Riemann solution contains the large time asymptotics for the solutions defined by a large class of Cauchy data (the elements of which are asymptotically or eventually constant on rays at large radii). Complete proofs of this statement for $d = 1$ can be found in [18,19]. Similarly the next to leading order behavior in the large time asymptotics is equivalent to Riemann solutions and Riemann problems with next to leading order scaling data included. From the known $d = 1$ theory, this next to leading order asymptotics is an N wave.

3. The Relation to Science

The conservation laws such as (1.1) arise in a number of problems in physics, fluid dynamics, chemistry and engineering. It is usually the case that the solution discontinuities are of special interest in these contexts. We take the basic and much considered case of the Euler equations for a polytropic gas in $d = 2$ space dimensions. There are scientific questions concerning which elementary waves actually arise in the solution to a given Riemann problem. Mathematical existence and uniqueness results would probably shed some light on these controversies. See [7] for a further discussion, where some favorable cases are mentioned for which such proofs might be tractable.

Even the co-dimension one Riemann problems do not have a theory which is adequate to meet the needs of science. Usually the Riemann solution is the first piece of analysis obtained by an engineer working on a problem which can be modelled by a nonlinear conservation law. The eigenvalues λ_j and the eigenvectors e_j define a geometry and topology in the state space, and it should be possible to read off some of the properties of the elementary waves and their interactions from these considerations. Such a conceptual understanding of the Riemann problem in the large would be most useful.

4. The Possibilities for a Mathematical Theory

The Riemann solutions have one fewer independent variable than does a general solution and the elementary waves have two fewer. This results in a considerable simplification, and means that the chances for an existence theory for Riemann solutions or elementary waves is much higher than it is for general data. For example the co-dimension three elementary waves are discussed in [8], and lead to transonic equations on the two sphere S^2, or a reduction from four to two independent variables. Because Sobolev and related inequalities are more favorable in low dimensions, there is an existence theory for hyperbolic systems of conservation laws in two independent variables [5]. On this basis we believe that progress can be made with co-dimension three elementary waves.

The codimension two elementary waves for the Euler equations of a polytropic gas were classified [7]. Five basic types of elementary waves were found. As with the co-dimension one Riemann problem, the differential equations in this case reduce to equations on the circle S^1 and have only one independent variable. Thus the differential equations present no problems from the point of view of local existence theory and the difficulty is transfered to the need for a global understanding of the wave curves and their geometry and topology in the state space.

For the co-dimension one Riemann problem there is the theory of Lax [15] for small data and the theory of Oleinik [21] for scalar equations. There are solutions known in the large for certain special cases [11,12,14,25]. Recent developments

[13] suggest that there is an important range of new phenomena, yet to be found in this area. We regard the co-dimension one Riemann problems, considered in the large, and the co-dimension two elementary waves as very promising problems for mathematical progress.

5. Computational Methods

Because the elementary waves and Riemann solutions contain an exact description of the singularities present in a general solution, it is tempting to try to utilize this knowledge as part of a computational algorithm. Chorin [3] first realized that Riemann solvers could be used in practical numerical solutions based on the random choice method [5]. Riemann solvers occur in a number of other algorithms [1,22,24,27] for numerical solution of systems of conservations laws. This point of view has received its furthest development in the method of front tracking [2,6,7,8], whereby a moving co-dimension j grid is used for each co-dimension j elementary wave in the solution, or at least for each of those of sufficient importance to warrant special consideration. Across this co-dimension j grid, the solution is discontinuous, and the weak derivatives which occur in the equation (1.1) are replaced by their correct analytic form in terms of jump conditions. In this way, finite difference quotients are never formed across jump discontinuities.

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